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Kalmár's Argument Against the Plausibility of Church's Thesis

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Abstract

In his famous paper, *An Unsolvable Problem of Elementary Number Theory*, Alonzo Church (1936) identified the intuitive notion of *effective calculability* with the mathematically precise notion of *recursiveness*. This proposal, known as Church's Thesis, has been widely accepted. Only a few papers have been written against it. One of these is László Kalmár's *An Argument Against the Plausibility of Church's Thesis* from 1959. The aim of this paper is to present Kalmár's argument and to fill in missing details based on his general philosophical thoughts on mathematics.

Introduction

In his famous paper *An Unsolvable Problem of Elementary Number Theory* Alonzo Church (1936) identified the intuitive notion of *effective calculability* with the mathematically precise notion of *general recursiveness*. This proposal is known as Church's Thesis. László Kalmár argued against the plausibility of this thesis at the International Colloquium "Constructivity in Mathematics" in Amsterdam in 1957, having been invited by Heyting. A short paper based on his talk was published in the conference proceedings in 1959.

The aim of this paper is to present Kalmár's argument and to fill in missing details by using Kalmár's interpretation of the incompleteness and undecidability results. These details are based on his papers written mostly in Hungarian about these issues, with his views on the philosophy of mathematics serving as a background.

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Section 1 gives a short survey of Kalmár's general views on the philosophy of mathematics. Then, Section 2 follows Kalmár's (1959) closely, while Section 3 provides the missing details. The *Appendix* contains some records of interactions between Church and Kalmár, though not directly relevant to the context.

1 Kalmár's General Views on Mathematics

Before turning to Kalmár's argument against the plausibility of Church's Thesis, I give a short introduction to his general views on mathematics. When discussing his argument in detail and attempting to make it more appealing by filling in missing details, I will rely on these general views.

Kalmár's main papers concerning the philosophy of mathematics, beyond his (1959), are *The Development of Mathematical Rigor from Intuition to Axiomatic Method* from 1942,[1] and *Foundations of Mathematics – Whither Now?*[2] from 1967. My remarks are based on these as well as on his paper on the epistemology of science, *On the Problem of the Foundation of our Knowledge* from 1965. Kalmár's main philosophical theses about mathematics and science are summarized in the following, necessarily intertwined, key points:

(1) *"[I]n mathematics there is no* ignorabimus.*"* (Hilbert 1900, p. 1102; and 1929, p. 233) Although Kalmár first learned about mathematical logic and understood the aims of Hilbert's proof theory from von Neumann's (1927), meeting Hilbert was decisive for Kalmár's career in choosing mathematical logic as his main focus. In 1928, Kalmár attended Hilbert's address at the International Congress of Mathematicians in Bologna (Hilbert 1929) and spent the Summer Term of 1929 in Göttingen.[3] There he attended Hilbert's course on Set Theory, the last third of which was devoted to mathematical logic and proof theory.[4] As a consequence Kalmár became one of the first advocates of Hilbert's metamathematical approach and his proof theory in Hungary.[5] Kalmár's philosophical views were also deeply influenced, as his (1942) clearly shows the impact

[1]Originally in Hungarian, translated to English in 2011.

[2]This paper is based on a talk Kalmár gave at the International Colloquium in the Philosophy of Science, held at Bedford College, London, in 1965, where he was invited by Imre Lakatos. It is of historical interest that although the traces of Lakatos' quasi-empirical view of mathematics can be found already around 1959 and 1961 in his unpublished essay *What Does a Mathematical Proof Prove?* (1978b), he introduced his view explicitly and first drew the distinction between quasi-Euclidean and quasi-empirical theories in the discussion after Kalmár's talk, entitled as *The Renaissance of Empiricism in the Recent Philosophy of Mathematics* (1967). According to Worrall and Currie, "Lakatos expanded these remarks into a longer paper which he completed in 1967. However, he withheld it from publication, intending to improve it further." (1978, p. 24) The extended essay, with the same title as his comments, appeared posthumously as (1978a). For further details on Kalmár's influence on Lakatos' philosophy, see Gurka (2006) and Máté (2008).

[3]Tóth (1976) and Szabó (2003). Kalmár mentions his visit in Göttingen in his Curricula Vitae from 1939, 1951, and 1976 (Szabó 2005, p. 2, p. 5, and p. 19 respectively).

[4]Gerhard Gentzen and Lothar Collatz were also enrolled in the same course. (Sieg 2013, p. 176)

[5]The first mathematical account of Hilbert's axiomatic method and proof theory in Hungarian is Kalmár's (1941).

of Hilbert and Bernays' (1934).[6] From early on, Kalmár shared Hilbert's deep conviction of the non-existence of the *ignorabimus,* that to every well-formulated problem a solution always can be found. He endorsed the axiomatic and meta-mathematical approach, but saw the strictly formal axiomatic level as a mere tool in mathematics and not as its sole purpose; just as Hilbert did.

(2) *There is no end to the development of mathematics.* By this, Kalmár not only means that we always will be able to prove new theorems, but that our methods will develop as well. That is, new rules of inference and methods of proof will be accepted and used in the future, leading to solutions of problems that were previously unsolvable. As a consequence of this stronger understanding of development, mathematics cannot have a fixed foundation once and for all, as these new methods have to be accommodated as well.

In his (1965) Kalmár argued for the broader claim that there is no final foundation for our scientific knowledge either. According to him, similar to the case of mathematics, we should acknowledge the endlessness of scientific development and should base scientific practice on this fact instead of looking for an ultimate foundation. However, this does not mean that there are inaccessible facts. On the contrary, Kalmár believed that during the endless development of the sciences and mathematics every problem will become solvable at some point. In the case of mathematics this statement coincides with Hilbert's conviction about the non-existence of an *ignorabimus*, as mentioned above.

(3) *Mathematics stems from empirical facts and its justification is (or should be), to some extent, empirical as well.* According to Kalmár, the mathematical objects, rules of inference and axioms originate in our physical experience and are "abstracted more or less directly from empirical facts" (1967, p. 192). Intuitions of mathematical objects, like point, line, plane are rooted in our spatial experience; numbers and operations on numbers are abstracted from counting and organizing physical objects.

As to the empirical justification, Kalmár is much more vague. He only has a few examples to support this claim, as the prevalent understanding of mathematics as a pure deductive science goes contrary to his suggestion of mathematics being "a science in need of an empirical foundation" (p. 193). He claims, based on historical examples, that mathematical practices are accepted by the community as long as they do not turn out to be fallacious. Furthermore that "the consistency of most of our formal systems is an empirical fact" (p. 192). However, on his view even the most abstract theorems of set-theory and mathematical logic "refer, presumably rather indirectly, to some properties of the real world, to which they are connected by a complicated chain of abstractions." (p. 206) This connection might allow for some kind of an empirical justification

[6]Beyond Hilbert and Bernays' (1934), it is strongly influenced by Sándor Karácsony, a Hungarian educationist and linguist (Szabó 2013). Kalmár belonged to the Exodus Circle, an intellectual circle led by Karácsony. Among the members we also find the mathematicians Rózsa Péter and Tibor Szele, the educationist Tamás Varga, and philosopher of mathematics Imre Lakatos, who had Karácsony as one of his opponents at his doctoral defense at the University of Debrecen in 1947. On the relationship and influences between Karácsony, Kalmár, and Lakatos see Gurka's (2006) and footnote 2. On Lakatos' Hungarian dissertation see Kutrovátz's (2002).

as well.

This picture of Kalmár's also explains the applicability of mathematics to physical reality. For, in this picture mathematics is not a purely abstract deductive science, but one that has strong empirical roots. Furthermore, even its confirmation is partly empirical (which, according to Kalmár, should be further stressed and explored).

(4) *Mathematics is always "done" in an intuitive, informal way*, not in one or another of several fixed formal frameworks. In Kalmár's account our basic mathematical concepts, like point, line, numbers etc., are "associated with a vivid, transparent [inner] picture" (1942, p. 271). Of course these intuitive pictures are rooted in our empirical experience and are, consequently, subjective and might differ from person to person. Intersubjectivity in mathematics is achieved through the formalization and axiomatization of these concepts. However, as Kalmár argues insistently, even the most abstract concepts in mathematics do have some kind of intuitive picture associated with them. These intuitive pictures and informal ideas concerning them are what guides the practicing mathematician. Though formal frameworks are crucial and indispensable in sharpening their ideas, they are always applied afterwards.

2 Kalmár's Argument Against the Plausibility of Church's Thesis

The aim of this section is to summarize Kalmár's argument based on his (1959). His argument contains a few peculiar points, such as questioning the objectivity of the notion "uniformity," "proof by arbitrary but correct means" and an unusually open-ended understanding of the notion of effective calculability. These notions and the ideas behind them will be explained in detail in the next section, *The Missing Details*.

At the outset of his (1959) Kalmár presents his stance on the status of Church's Thesis and states his goals as follows:

> In the present contribution I shall not disprove Church's Thesis. Church's Thesis is not a mathematical theorem which can be proved or disproved in the exact mathematical sense, for it states the identity of two notions only one of which is mathematically defined while the other is used by mathematicians without exact definition. (p. 72)

So Church's Thesis is not a theorem, but, Kalmár claims, it should not be taken as a definition either. For if it would be regarded as a definition, there would be a chance that in the future someone would be able to define a function which is not effectively calculable in Church's sense, yet it could be effectively calculated for any given arguments. Hence Church's Thesis is possibly refutable while definitions are not. Furthermore Kalmár (1957a, p. 28) agrees with Post that the identification of the intuitive notion of effective calculability with general recursiveness is a "working hypothesis" and "to mask this identification under a definition hides [...] the need of its continual verification."[7] (Post 1936, p. 105, footnote 8) Thus, according to Kalmár:

> For this reason, it seems [to] me better to regard such statements as Church's Thesis [...][8] as propositions[9] rather than definitions, however, not mathematical but "pre-mathematical" ones. (Kalmár 1959, p. 73)

In the abstract of his talk, Kalmár described "pre-mathematics" that "discusses such questions as plausibility of hypotheses, adequacy of definitions etc. to be used in mathematical reasoning." (1957c) Thus Church's Thesis is a "pre-mathematical" proposition as the question is whether it is *plausible* or not, since it cannot be mathematically proved or disproved.

It is important to note that in considering Church's Thesis as a "pre-mathematical proposition," for Kalmár the emphasis is on its "pre-mathematical" status. That is, the arguments for or against it should consider its plausibility and cannot be purely mathematical ones. With respect to the "proposition" or "definition" distinction, he himself points out that those are manifestations of only slightly differing perspectives. For, addressing the "plausibility of a pre-mathematical proposition" or the "pre-mathematical adequacy of a formal definition" are rather similar questions. Indeed, Kalmár remarks that:

> The more than two pages of Church's paper (1936) filled with plausibility (hence pre-mathematical) arguments for his thesis, show that his opinion about this question does not differ much from mine.[10] (p. 73)

As a consequence, Kalmár sees his own argument against the plausibility of Church's Thesis to be "likewise pre-mathematical" (p. 73).

Although his talk was presented at the Colloquium on "Constructivity in Mathematics," he added the following methodological remark before turning to his argument:

> In these arguments, I shall freely use the *tertium non datur* [law of excluded middle], hence, they do not claim to be accepted by

[7] However, as we will see later in detail, Kalmár disagrees with Post in interpreting Church's undecidability results as a "fundamental discovery in the limitations of the mathematicizing power of Homo Sapiens" (Post 1936, p. 105, footnote 8).

[8] As Kalmár remarks, the same holds for the identification of solvable problems with the sets defined with one parameter whose characteristic function is recursive.

[9] As propositions are either true or false, they can be refuted.

[10] Kalmár refers to §7, *The notion of effective calculability*, of Church's (1936, pp. 356–358). There Church introduces his thesis with the following remark:

> We now define the notion, already discussed, of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers. This definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion. (p. 356)

> adherents of constructivistic doctrines which reject the *tertium non datur*. (p. 73)

Footnote 14 of this paper points to the uses of the law of excluded middle in his argument. So why did Kalmár present these ideas at this forum? Again, Kalmár attended the Colloquium on Heyting's invitation. Furthermore, in a letter of 23 January 1957 Kalmár turned to the Mathematical Division of the Hungarian Academy of Sciences to request travel funding for the Colloquium. There he indicates that he would like to present his argument against Church's Thesis, "which hypothesis concerns the identification of *constructively definable functions*, i.e. such arithmetical functions whose value can be calculated in a finite number of steps for any given argument, with that of general recursive functions." (1957b, translation and italics by me) This understanding of Church's Thesis was not uncommon during the 1950s and 1960s and made Kalmár's talk appropriate for the Colloquium. (Even though Kalmár uses the law of excluded middle, nevertheless, some feel that his argument shares traits with intuitionistic/constructivist approaches. These similarities and differences are not directly relevant for his argument, but they do contribute to a deeper understanding of Kalmár's views. These issues are discussed at the end of *The Missing Details* section.)

Kalmár's objection to the plausibility of the thesis is based on his belief that not every effectively calculable function is recursive, i.e. the set of effectively calculable functions is broader than that of recursive ones.[11] Having said that, he immediately adds the following clarificatory remark:

> We regard as effectively calculable any arithmetical function, the value of which can be effectively calculated for any given arguments in a finite number of steps, irrespective how these steps are and how they depend on the arguments for which the function value is to be calculated. In particular, I do not suppose the calculation method to be "uniform". (p. 73)

Moreover, Kalmár doubts that "uniformity" has an objective meaning. For, let us imagine a school-boy for whom "the method for the solution of the diverse arithmetical problems he has to solve does not seem uniform until he learns to solve equations." (p. 73) Before group theory was discovered, mathematicians were in the place of the school-boy as they used methods in algebra, geometry and number theory which are now considered as group-theoretic methods. A more detailed explanation of Kalmár's point will be given in the next section.

In order to point to some unplausible[12] consequences of Church's Thesis, Kalmár considers a non-recursive function of the form:

[11] At the same conference Rózsa Péter argued that not every recursive function can be regarded as effectively calculable (Péter 1959).

[12] Kalmár's use of words.

$$\psi(x) = \mu_y(\varphi(x,y) = 0) = \begin{cases} \text{the} & \text{least natural number } y \text{ for which} \\ & \varphi(x,y) = 0 \text{ if there is such a } y, \\ & \\ 0 & \text{if there is no natural number } y \text{ such} \\ & \text{that } \varphi(x,y) = 0 \end{cases}$$

where φ is a recursive function of two arguments. Kleene has shown that such functions exist.[13]

For any natural number p for which a natural number y exists, such that $\varphi(p,y) = 0$, we can calculate the value of $\psi(p)$ in finitely many steps by successively calculating $\varphi(p,0), \varphi(p,1), \varphi(p,2), ...$ as φ is recursive.

> On the other hand, for any natural number p for which we can prove, not in the frame of some fixed postulate system but by means of arbitrary – of course, correct – arguments that no natural number y with $\varphi(p,y) = 0$ exists, we have also a method to calculate the value $\psi(p)$ in a finite number of steps: prove that no natural number y with $\varphi(p,y) = 0$ exists, which requires in any case but a finite number of steps, and gives immediately the value $\psi(p) = 0$. (p. 74)

Now, if based on Church's Thesis we suppose that ψ is not effectively calculable as it is non-recursive, we can infer that none of the above two methods will suffice to calculate it everywhere. Hence we are led to:[14]

> the existence of a natural number p for which, on the one hand, *there is no natural number y such that* $\varphi(p,y) = 0$, on the other hand, *this fact cannot be proved by any correct means.* (p. 74)

That is, according to Kalmár, "a consequence of Church's Thesis which seems very unplausible." (p. 74)

Now if we consider the proposition $\exists y(\varphi(p,y) = 0)$ with the p above, we can recognize that it is an "absolutely undecidable proposition". It is undecidable, as it cannot be proved, because it is false, but the negation of the proposition cannot be proved either, since such a proof would give the value 0 for $\psi(p)$. According to Kalmár its undecidability is "absolute" in the following sense:

> As a matter of fact, the problem, if the proposition in question holds or not, does not contain any parameter and, supposing Church's Thesis, *the proposition itself can be neither proved nor disproved,* not only in the frame of a fixed postulate system, but *even admitting any correct means.* (p. 75)

Kalmár emphasizes that his "absolutely undecidable proposition" differs both from Church's and Gödel's examples. It is neither a problem with a parameter as

[13] Kalmár cites Theorem XIV. of Kleene's *General Recursive Functions of Natural Numbers* (1936), which states that the function $\varepsilon y[T_1(x,x,y)]$ is non-recursive.

[14] Here Kalmár remarks that the law of excluded middle is used and also points to the definition of ψ where it has been utilized already.

in Church's case nor is it an undecidable proposition *relative* to a fixed postulate system as in Gödel's.

"However," continues Kalmár, "this 'absolutely undecidable proposition' has a defect of beauty: we can decide it, for we know, it is false." Thus:

> *Church's Thesis implies the existence of an absolutely undecidable proposition which can be decided.* (p. 75)

That is a "a very strange consequence indeed." (p. 75)

Furthermore, the absolute undecidability of this proposition cannot be proved "by any correct means." (p. 75) Since a proof of its undecidability would show that there is no such y for which $\varphi(p, y) = 0$ holds, which at the same time would prove $\neg\exists y(\varphi(p, y) = 0)$ and entail that $\psi(p) = 0$. Kalmár concludes:

> The fact that some consequences of Church's Thesis cannot be proved by any correct means can be regarded, I think, as arguments against its plausibility. (p. 76)

At the end of his talk[15] Kalmár summarizes his opinion as follows:

> There are pre-mathematical concepts which must remain pre-mathematical ones, for they cannot permit any restriction imposed by an exact mathematical definition. Among these belong, I am convinced, such concepts as that of effective calculability, or of solvability, or of provability by arbitrary correct means, the extension of which cannot cease to change during the development of Mathematics. (p. 79)

Thus, Kalmár sees effective calculability as an open ended notion that should not be restricted, as it will change with the development of mathematics.

*   *   *

We have to be careful how we interpret Kalmár's argument. Many scholars took him to be trying to refute Church's Thesis. However, he was deliberate in phrasing it only as an argument against the *plausibility* of the Thesis and nowhere did he indicate that he refuted or even attempt to refute Church's Thesis. Indeed, in his (1967) he describes the upshot of his argument in this way:

> I pointed out that plausible arguments can be given *against* Church's Thesis as well as for it. (p. 193, footnote 1)

[15] In the intermediary pages 77-79 he gives a slightly more formal version of his argument where he tries to replace the "vague concept of a proof by arbitrary means [...] by a more definite one." (p. 77) But on page 79 he admits that "the new form of my argument is as heuristic as the old one."

And he continues with the following remark that is rather close to the general view:

> But I have no objection against Church's Thesis if it is taken as an *empirical* one, confirmed several times in practice, but like any other empirical thesis, to be abandoned if a counter-example is found in the future. (p. 193, footnote 1)

This more permissive attitude can be seen also where Kalmár talks about possible refutations of the Thesis later:

> I consider any proposition stating the absolute unsolvability of some problem (with parameter) proved on the basis of Church's Thesis, as a potential falsifier of all theories which are based on this thesis, of course without being sure or even without suggesting that any of them will be falsified, by solving the problem in question using some acceptable (thought of course, not recursive) method, some time in the future. (p. 207)

A similar understanding of Church's Thesis can be found already in Kalmár's (1959). The Thesis is said to be a "working hypothesis" (p. 72) and later it is characterized:

> as a challenge,[16] to find, instead of the class of general recursive functions, either a less inclusive class which cannot be shown to exclude some function which ought reasonably to be allowed as effectively calculable, or a more inclusive class which cannot be shown to include some arithmetical function which cannot be seen to be effectively calculable. (p. 76)

Of course Kalmár believed that there are effectively calculable functions which are not general recursive.[17]

Németi and Andréka (2006) mention that this is how Kalmár presented the thesis to his students as well. Namely as a challenge that should "tease" and invite scholars of future generations to attempt to refute it.

[16]Kalmár borrowed this phrasing from Church's (1938) where Church writes the following about his definition of the constructive second number class:

> It is my present belief that the definition is absolute in this way–towards those who do not find this convincing the definition may perhaps be allowed to stand as a *challenge*, to find either a less inclusive definition which cannot be shown to exclude some ordinal which ought reasonably to be allowed as constructive, or a more inclusive definition which cannot be shown to include some ordinal of the second class which cannot be seen to be constructive. (p. 224, italics by me)

[17]Kalmár's argument can be seen as a proposal of such a more inclusive class, namely "by subjoining all arithmetical functions ψ defined by an equation of the form $\psi(x) = \mu_y(\varphi(x, y) = 0)$ with a general recursive function φ of two arguments to the class of the general recursive functions" (p. 76).

3 The Missing Details

At first sight, Kalmár's argument contains a few peculiar, ambiguous or vague ideas, such as questioning the objectivity of the notion of uniformity, "proof by arbitrary but correct means," and the above, open-ended understanding of effective calculability. I believe that these ideas can be better understood if we take into account his general views concerning mathematics. In addition, I will allude to some of his other papers on unsolvable problems, written mainly in Hungarian, to present his argument more fully.

Questioning the objectivity or meaningfulness of uniformity and dismissing it as a requirement for effective calculability is the first peculiar point. Indeed, as Moschovakis points out in his review:[18]

> It is customary not to call a procedure for computing the values of a function effective, unless it is *uniform* for all arguments. (1968, p. 472)

As an explanation, let us recall Kalmár's school-boy metaphor. He remarks that before group theory was discovered, even though mathematicians used methods in algebra, geometry and number theory which are now considered as group-theoretic methods, we were in the place of the school-boy. He also believes in the endless development of mathematics, that is, that new rules of inference and methods of proof will be accepted and used in the future. This means that, in some sense, we are and always will be in the school-boy's place. In his (1957a, p. 37) Kalmár elaborated more on this issue. There he said that being in the position of the school-boy shows that the notion of uniformity is always relative to our level of mathematical development at a given time. It is this relativity that is contrasted with the usually assumed objectivity of uniformity. Kalmár says that the notion of effective calculability and Church's Thesis can only have an objective meaning, if they do not involve the relative notion of uniformity.

The most ambiguous or vague notion in the argument is "proof by arbitrary means" or as it first appears:

> [a proof] not in the frame of some fixed postulate system but by means of arbitrary – of course, correct – arguments (1959, p. 74)

Understanding this notion is crucial because of its frequent recurrence and the weight it carries in Kalmár's argument.[19]

[18]Moschovakis reviewed three arguments against Church's thesis: Kalmár's (1959), Péter's (1959), and Porte's (1960). He also reviewed Mendelson's (1963) which is likewise a review of these papers. For further arguments against the thesis see Bowie's (1973) where he claims that while the notion of computability is intensional, the notion of recursiveness is extensional, and its discussion by Ross (1974) and Berg and Chihara (1975).

[19]Webb pointed out (1980, pp. 189-190) that Gödel discussed a similar notion of proof in the *Introduction* of his dissertation (1929). Gödel noted that his completeness theorem, making use the law of excluded middle (for countable collections), might be seen as a positive solution to the Entscheidunsproblem: "For what is to be proved can, after all, be viewed as a kind of decidability (every expansion of the restricted functional calculus either can be recognized as valid through finitely many inferences or its validity can be refuted by a counterexample)."

We could proceed again from the endless development of mathematics, but now we can go into finer details. First of all, the frame of a fixed postulate system is rejected. We should recall how Kalmár thinks about mathematical practice. When we "do" mathematics we "do" it at an intuitive level, not in a formalized framework. Thus, we do not fix one (or many) formal frameworks at the outset and "do" mathematics in them afterwards. First we try to prove statements at an intuitive level and then justify the correctness of the proof. Formalization is a tool in such a justification, which is at the same time partly an empirical issue for Kalmár.

To proceed further, let us go back to Moschovakis' review. He draws the following conclusion:

> Kalmár's "correct means of proof" must include new intuitions about truth in set theory. It seems to this reviewer that Kalmár's argument simply shows that the procedure "try to prove by all correct means..." is not effective.[20] (1968, p. 472)

Kalmár would accept and be satisfied (to a large extent) with this judgment, if effectiveness is understood in Church's sense and not his. For, as already quoted above, when Kalmár talked about possible refutations of Church's Thesis he said:

> I consider any proposition stating the absolute unsolvability of some problem (with parameter) proved on the basis of Church's Thesis,

According to Gödel some might claim that the use of the law of excluded middle renders his proof worthless, as: "On the other hand, the principle of the excluded middle seems to express nothing other than the decidability of every problem." (p. 63) Gödel provides two arguments against this understanding. First, that this argument relies on the intuitionist interpretation of the law of excluded middle. Second, he emphasizes that: "Even if we accept this interpretation, what is affirmed is the solvability not at all through specified means but only through all means that are *in any way imaginable*, while what is shown is precisely that every valid expression can be derived through completely *specified, concretely enumerated* inference rules." (pp. 63-65) Gödel not only contrasts the completely specified means of proof with proofs by "all means that are in any way imaginable", but in footnote 4 remarks that "It seems questionable however, whether a notion of solvability that is so sweeping [...] makes any sense at all."

[20]Mendelson (1963, p. 203) and Nelson (1987, p. 594) raised similar concerns about the "effective enumerability" and "algorithmic character" of such arbitrary but correct means of proofs, respectively.

Kleene in his recollection (1987) went further:

> I was present at the Amsterdam Colloquium of 1957 when my good friend László Kalmár presented his argument against the plausibility of Church's thesis; I immediately concluded, as fast as I heard it, that he had not given an effective procedure for deciding as to the truth or falsity of $(x)\overline{T}(n,n,x)$. He would not be able to tell me in advance in a finite communication (no matter how long we both should live) what set of atomic rules would completely govern the concrete steps in his search for proofs by "arbitrary correct means" of $(x)\overline{T}(n,n,x)$. I refrained from embarrassing him at the Colloquium by asking him for them on the spot. (pp. 494-495)

In his *Introduction to Metamathematics* Kleene used essentially the same function as Kalmár to provide an example of a function that is *not* effectively calculable (1952, pp. 317-318).

> as a potential falsifier of all theories which are based on this thesis, of course without being sure or even without suggesting that any of them will be falsified, by solving the problem in question using some *acceptable (thought of course, not recursive) method*, some time in the future. (1967, p. 207, italics by me)

Kalmár's view is that if such methods are found, they should be considered as 'effective' as well. In order to support this claim, let us take a closer look at how Kalmár regarded the question of unsolvable problems.

In 1949, Kalmár was elected as a corresponding member of The Hungarian Academy of Sciences. In his inaugural lecture he gave a simpler and shorter proof of Post's and Markov's result on the unsolvability of Thue's problem (1952a). In 1951, Kalmár summarized his talk in a letter to Tibor Szele. After defining Thue's problem, he wrote the following on finite decision procedures:

> Every mathematician "feels" what we understand by a procedure, with which we can decide a problem in finitely many steps. This feeling is sufficient to determine whether one accepts a given proposal of such a method as a finite decision procedure or not. But in order to prove a negative statement, that is, a statement of non-existence of such a method, we have to define precisely what we understand by a finite decision procedure. [...] A few people worked on an accurate definition of finite decision procedure and gave different definitions, which more or less plausibly cover what one understands by a finite decision procedure. (Szabó 2005, Letter no. 227, pp. 324–325, translation by me)

Later in the letter Kalmár assessed Church's notion of λ-definability as "the least plausible" characterization of finite decision procedures. In 1936, Church stated his thesis in terms of an equivalent notion, namely of (general) recursive functions.

Thus, for Kalmár, the acceptance of Church's Thesis is more than just the acceptance of the identification of the intuitive notion of effective calculability with a precisely defined class of functions; it is, at the same time, the acceptance of this precise characterization as a restriction of the notion of finite decision procedures. Recalling Kalmár's general view on mathematics, this is what he disapproved of in the first place. As he emphasized at the end of his talk in Amsterdam, this pre-mathematical notion "must remain pre-mathematical" and its "extension [...] cannot cease to change during the development of Mathematics." (1959, p. 79)

To understand Kalmár's motivation for his open ended understanding of effective calculability, we have to take a look at his interpretation of Gödel's Incompleteness Theorems and Church's Undecidability Theorem, and his critique of certain interpretations of those results. Most of Kalmár's papers on these issues date from the late 1940s and the 1950s. After the Second World War until his death Kalmár held dialectical materialist views. Thus, these papers are phrased in a politically laden vocabulary. However, as I show, the ideas that can be extracted from these papers can be traced back to Kalmár's general philosophical views on mathematics and the sciences. In the following pages I will first quote Kalmár's politically laden statements and then explain their pure philosophical content.

In Kalmár's exposition dialectical materialism tells us that we can come to understand and learn everything about objective reality through an endless progression (1952b and 1957a). By "objective reality" Kalmár only means the 'physical world'.[21] Although it might not be evident at first, this characterization of dialectical materialism has strong implications for mathematics as well. Because for Kalmár, mathematics always stems from empirical facts through experience and even its justification is, at least in part, empirical. Thus, mathematics is always connected, at least indirectly, to "objective reality." Hence Kalmár's exposition of dialectical materialism amounts to the belief, mentioned among his general philosophical views, that during the endless development of the sciences and mathematics every problem will become solvable at some point.

Indeed, in his (1959) Kalmár mentions, and in his (1957a) he emphasizes that in his argument against the plausibility of Church's Thesis the function φ can be chosen to be an elementary function.[22] It is important for him so he can claim that these arguments are particularly closely connected with "objective reality." For, "every proposition [such as $\varphi(p, y) = 0$] that contains only non-negative integers and the basic arithmetic operations [addition, subtraction, multiplication and division] refers to quantitative relations in objective reality." (1957a, p. 34) Kalmár even describes how these operations emerged through our experience from organizing physical objects.

In 1956 Kalmár gave a talk on Church's Thesis and Unsolvable Mathematical Problems at the Hungarian Academy of Sciences (1957a). In this talk he mentioned two interpretations of Gödel's Incompleteness Theorems and Church's Theorem which, according to him, "show the signs of agnosticism." Any interpretation that understood these results as arguments for limits of possible human knowledge were labeled by Kalmár as agnostic arguments from idealist philosophers (1952b, p. 94 and 1957a, p. 19 and 28). For, these interpretations imply that there are unknowable features of, or unsolvable problems about "objective reality." Hence these interpretations are in clear conflict with Kalmár's above understanding of dialectical materialism.

The first such interpretation quoted by him is footnote 8 of Emil Post's *Finite Combinatory Processes – Formulation 1* (1936) claiming that Church's Theorem is:

> a fundamental discovery in the limitations of the mathematicizing

[21]In his (1965) he uses it interchangeably with "world" and "material world," and in his (1974, p. 492) in a similar place he uses a Hungarian expression that can be translated as "real world."

[22]Kalmár defined the notion of elementary functions in his (1943, pp. 2-3). These functions are built up from the constant 1 and integer variables and finitely many applications of the following operations: addition, (truncated) subtraction, multiplication, (floored) division, and bounded summation and production.

> power of Homo Sapiens. (p. 105)[23]

The second is a review from Wedberg (1950) where Wedberg says that:

> The non-existence of an *ignorabimus* in mathematics can, of course, no longer be maintained in the form which appeared plausible before the well-known results of Gödel and Church. (p. 246)

Indeed, the acceptance of an ignorabimus amounts to the acceptance of the existence of a mathematical statement that is neither provable nor disprovable. Something that goes clearly against Kalmár's deepest convictions. Not only to his dialectical materialist beliefs, but in his belief in the non-existence of the ignorabimus as well.

However, according to Kalmár, Gödel's Theorems are actually in accord with the epistemology of dialectical materialism, since those only show that we always have to extend our mathematical systems and improve our methods if we want to solve "every mathematical problem." That is, the belief that we can come to understand and learn everything about objective reality through an endless progression is not contradicted by the endless need to extend our mathematical systems.

Kalmár called also attention to the fact that although Church's Theorem is considered a sharpening of Gödel's results as it provides an "absolutely undecidable" problem in contrast to a "relatively undecidable" problem, it is a problem with a parameter, i.e. an infinite set of problems. Consequently the non-existence of a uniform decision procedure does not provide an absolutely undecidable proposition (in Gödel's sense). It only means that we have to decide each proposition one after another with distinct methods. Thus, Church's Theorem does not support agnosticism either (1952b and 1957a).

Kalmár provided another, and in his opinion stronger, argument against "agnostic" interpretations of Church's Theorem in particular. In most cases they were phrased aiming at different mechanistic views that he actually had no intention to defend. Nevertheless, he felt compelled to respond to such interpretations in detail because with slight modifications they could clearly be seen as the strongest threats to his version of dialectical materialism. In this context Church's Theorem was seen as having much stronger philosophical consequences than those of Gödel's.

In 1948, at the 10th International Congress of Philosophy he sketched a proof, by further developing some unpublished ideas and results of Rózsa Péter, to show that Gödel's Theorem can be so generally formulated that Church's Theorem becomes a "consequence, or even a particular case" of it.[24] (1949, p. 758) Kalmár regarded this as a strong argument against the "agnostic" interpretations of Church's Theorem, because "it is impossible that a particular case

[23] Kalmár quoted Post's paper in this regard in his talk in Amsterdam in 1957 as well (1959, p. 73, footnote 1). However, Kalmár agrees with Post about "the need of its [Church's Thesis] continual verification." (p. 105)

[24] Neither Péter nor Kalmár did publish these results in detail. A complete paper of their joint work is assembled based on Kalmár's manuscripts by Szabó (2015).

of a theorem should have stronger philosophical consequences than the theorem itself." (p. 758) Thus, it cannot be maintained that in contrast to "Gödel's Theorem furnishing but a relatively undecidable problem, Church's Theorem shows an absolute limit of our knowledge." (p. 758) For Kalmár, who was convinced that Gödel's results are in accord with the epistemology of dialectical materialism, this amounts to showing that Church's Theorem does not constitute a threat either. For him, it is a refutation of these "agnostic" claims.

Now let us go back to Kalmár's argument against the plausibility of Church's Thesis. If one accepts the Thesis, he claims, then one can infer "*the existence of an absolutely undecidable proposition which can be decided.*" (1959, p. 75) In Amsterdam he evaluated this as a "very unplausible" and "very strange consequence." In his talk at the Hungarian Academy of Sciences in 1956, where he presented the same argument, he added the following remark to his evaluation. The above consequence of Church's Thesis is "strongly agnostic" (1957a, p. 33), since it states:

> there is a property of objective reality that cannot be proved by any correct means. (1957a, p. 34, translation by me)

Thus, no one can accept Church's Thesis who believes that we can come to understand and know everything about objective reality (1957a, p. 34). To sum up, Kalmár disapproved of the agnostic interpretations of Church's Theorem and viewed Church's Thesis as "unplausible," in part, because it has an agnostic consequence.[25]

From this we can get a deeper understanding of Kalmár's view of effective calculability, which he exposed at the beginning of his talk:

> We regard as effectively calculable any arithmetical function, the value of which can be effectively calculated for any given arguments in a finite number of steps, irrespective how these steps are and how they depend on the arguments for which the function value is to be calculated. (1959, p. 73)

This view is motivated not only by his mathematical, but also by his (dialectical materialist) epistemological views. He is interested in any kind of knowledge we can acquire about "objective reality" with the aid of mathematics. Thus, according to Kalmár, every "correct" mathematical result obtained "in a finite number of steps" is obtained effectively. This understanding of effective calculability conceives of it as a notion "which cannot cease to change during the development of Mathematics", and makes Church's Thesis with the restricted notion unacceptable, as it leads, at least in Kalmár's reading, to agnostic consequences.

[25]The question might arise: "Why did Kalmár not mention his political or epistemological views in 1957 in Amsterdam?" In an interview (1972) he tells that after his talk in 1948, where he mentioned his views, the audience was more concerned about those (claiming that Kalmár "had to" argue for dialectical materialism, since he came from the other side of the Iron Curtain) than about the content of his arguments. I think this experience convinced him not to expose those views of his again in 1957.

However, Moschovakis pointed out correctly that this way of understanding effective calculability is substantially different from the usual one, where it is an attempt to capture "mechanical procedures." As a consequence, although Kalmár's argument is coherent based on his interpretation of effective calculability, it does not affect Church's Thesis as it is usually understood.

Kalmár, Gödel and the Ignorabimus

Hilbert, in his address to the International Congress of Mathematicians in 1900, talked about the deep conviction of mathematicians that to every well-formulated problem a solution can be found, or as he put famously, "in mathematics there is no *ignorabimus*." Kalmár, who spent the summer of 1929 in Göttingen, was deeply convinced about the veracity of this statement. From the 1940s he saw this statement as an integral part of his dialectical materialist views. It is interesting that Gödel, who called himself an antimaterialist, drew the same conclusion concerning his own results, namely, that one can still maintain the view that there is no ignorabimus.

In his unpublished draft of a conference talk (193?)[26] Gödel mentions that Hilbert was so firm in his belief in the non-existence of ignorabimus that "he even thought a mathematical proof could be given for it, at least in the domain of number theory." But the answer turned "out to be negative even in the domain of number theory" on account of Gödel's incompleteness results. This could mean, according to Gödel, two different things: (1) the problem "in its original formulation has a negative answer", or (2) through the formalization of evident axioms and evident inference rules we lost something. Interpreting his own results, he asserts: "It is easily seen that actually the second is the case, since the number-theoretic questions which are undecidable in a given formalism are always decidable by evident inferences not expressible in the given formalism". Thus, on the one hand, Hilbert's conviction remains untouched, and on the other hand we see that mathematical evidence cannot be formalized and as a consequence it "is not possible to mechanize mathematical reasoning." (p. 164)

Gödel's explanation of his results and some of Kalmár's arguments resemble each other remarkably. When they interpret the incompleteness and undecidability theorems they both argue that the conviction of the non-existence of the ignorabimus can be upheld. To see where Kalmár and Gödel part company, we take another look at Kalmár's argument against the plausibility of Church's Thesis. Here the order is turned upside down. Kalmár is not arguing for the non-existence of the ignorabimus anymore, but presupposes it and uses it to support his argument. He shows that based on *his* (completely unrestricted) understanding of effective calculability the existence of an absolutely undecidable proposition can be inferred. It is this conclusion that makes Church's Thesis unacceptable, if we are convinced that there is no ignorabimus. The key here is that effectiveness is in no way restricted in Kalmár's understanding. We saw

[26]The draft was published as (193?) in the third volume of Gödel's *Collected Works*. The questionmark results from the fact that the precise date and occasion of the talk are unknown.

that Gödel talked about the impossibility of the *mechanization* of mathematics, not about absolutely undecidable problems. Later in his (193?) Gödel makes it clear that by undecidability of a class of problems he means that "there exists no *mechanical procedure* for deciding every proposition in the class." (p. 165)

Kalmár's Relation to Intuitionism and/or Constructivism

Occasionally the question arises whether Kalmár held intuitionistic or constructivist views. The short answer is that he did not. However, seeing his argument in isolation gives some reasons to think otherwise. One reason that makes the question plausible is that Kalmár delivered his talk at the International Colloquium "Constructivity in Mathematics."

Moschovakis writes in his letter on March 15, 1968, to Alonzo Church accompanying his review (quoted in Sundholm's (2014, pp. 29–30)) that "Kalmár's argument [...] is rather close [...] to some of the arguments that Brouwer used in his later years concerning the so-called 'creating agent'." The formal version extracted from those arguments by Kripke is sometimes called Kripke's Schema.[27] Moschovakis is aware that Kalmár uses classical reasoning, as he says, "[i]n the end I decided that the explicit use of the classical ideas by Kalmár made his argument sufficiently different so I left the analogies with these ideas of Brouwer out of the review." Then he remarks that "it is interesting to note that within the kind of intuitionistic mathematics that accepts Kripke's Schema, Kalmár's algorithm is acceptable."[28] At the end of his letter he expresses his uncertainty about Kalmár's views as he acknowledges that "I do not know if Kalmár knew of the Brouwer ideas and whether he was influenced." As indicated by Church's response to Moschovakis (see below), Kalmár most likely was not influenced by Brouwer's ideas.

Another reason we might be led to believe Kalmár had intuitionistic beliefs is the resemblance of his informal approach to such concepts as effective calculability, solvability, or "provability by arbitrary correct means" with such intuitionistic approaches as the following from Michael Dummett:

> For the Hilbert school, and for formalists properly so called, formalization is integral to an exact treatment of mathematics; but the original impulse to formalization did not come from them, but from the logicists, for whom the formalization of a theory was a necessary means of identifying its basic principles, so that they could then show these to be derivable from pure logic. The *intuitionists*, on the other hand, were from the start *hostile to formalization*: for

[27]For a detailed and accessible description of Brouwer's ideas and Kripke's Schema see van Atten's (2008).

[28]For a discussion about the relation of Kripke's Schema to Kalmár's Argument and Church's Thesis see Webb's (1980, pp. 209-211) and Kreisel's (1970, p. 128 and footnotes 9 and 10 on p. 143). In footnote 9 Kreisel states that although Kripke established "the absurdity of proving Church's Thesis" it is "not enough for Kalmár's purpose" as he insisted on using classical logic "explicitly". And he adds: "It may fairly be said that (Kalmár 1959) does not provide a framework within which one might even *begin* to refute Church's Thesis."

> them, it is highly unlikely that the mental constructions intuitively recognizable as proving a statement of a given theory should be isomorphic to the formal proofs of any calculus, recognizable as such by a mechanical procedure making no appeal to meaning. There is therefore some irony in the intensive study that has been made by logicians of intuitionistic formal systems; but it can reasonably retorted that, just as Gödel's incompleteness results did not destroy the interest in investigating proof-theoretical questions relating to classical theories, so the fact that we *never expect to have a complete formalization of any intuitionistic theory* should not deter us from studying similar questions in this area. (1977, p. 300, italics by me)

In spite of these signs, Kalmár did not hold intuitionistic or constructivist views. In case of this particular paper, as was mentioned in the *Introduction*, he was invited to the Colloquium by Heyting. More importantly, he himself emphasizes in the paper that "I shall freely use the *tertium non datur* [law of excluded middle], hence, they do not claim to be accepted by adherents of constructivistic doctrines which reject the *tertium non datur*" (Kalmár 1959, p. 73) and later points to those occasions where it has been used.[29]

Church's response to Moschovakis on April 15, 1968 (quoted in Sundholm's (2014, pp. 30–31)) captures Kalmár's influences accurately:

> Though I have no definite information, I think it unlikely that Kalmár's argument was suggested or influenced by the ideas of Brouwer to which you refer. This is simply on the ground that Kalmár's mathematical publications have never shown any great concern with intuitionism. And certainly his argument in the paper under review, as you report it, is the very antithesis of intuitionism.[30] That is,

[29] However, Murawski in his (1999, p. 90) makes the following remark:

> Observe that one can treat the argumentation of Kalmár given above not as an argumentation against Church's thesis but as an argumentation against the law of excluded middle (*tertium non datur*) – this law played a crucial role in Kalmár's argumentation.

Kalmár and Markov also recognized this possible interpretation:

> Of course, from my above arguments other consequences can be drawn, if one wants to do so. For instance, one can insist upon Church's thesis and regard these arguments as quasi-refutations of the *tertium non datur.* So did Markov during the Third All-Union Mathematical Congress in Moscow 1956. (Kalmár 1959, p. 79, footnote 2)

[30] Interestingly, Kreisel in his review (1960) of Kalmár's argument states that the statement "that the notion of an effective rule must remain "pre-mathematical" and does not permit restrictions imposed by an exact mathematical definition [...] contradicts the very basis of intuitionist mathematics where the notion of construction is taken as a primitive (mathematical) notion, C[hurch's thesis] taking the form of a mathematical statement

$$(\forall\alpha)(\exists e)(\exists\beta)(\forall x)\,\{\alpha(x) = U[\beta(x)]\&T[e, x, \beta(x)]\}$$

with α ranging over constructive number theoretic functions."

> he assumes in effect that, for every x, either there is a proof of Ax (by some correct means) or there is a proof of the negation of Ax (by some correct means, in each case). And this is a special case of, and clearly kindred in spirit to, Hilbert's principle of the solvability of every mathematical problem, which Brouwer once so roundly denounced.

Indeed, Kalmár was strongly against intuitionism and constructivism. It is not surprising if we think of his strong commitment that such concepts as "provability by arbitrary correct means" "cannot permit *any* restriction imposed by an exact mathematical definition." (p. 79, italics by me) Also, Church is right, as we saw in the previous subsection, that his philosophy was deeply influenced by Hilbert in many respects, who was famously against intuitionism:

> What Weyl and Brouwer do amounts in principle to following the erstwhile path of Kronecker: they seek to ground mathematics by throwing overboard all phenomena that make them uneasy and by establishing a dictatorship of prohibitions *à la* Kronecker. But this means to dismember and mutilate our science, and if we follow such reformers, we run danger of losing a large number of our most valuable treasures. (Hilbert 1922, p. 1119)

Indicative of the same attitude, one of Kalmár's letters contains information about talk of his in Debrecen in 1953 under the similarly phrased title *Critique of attempts to mutilate mathematics* (translation by me) where the abstract specifically mentions the intuitionism of Weyl and Brouwer and constructivism of Lorenzen as its targets. (Szabó 2005, Letter no. 242, p. 341)

Appendix: Church and Kalmár

This *Appendix* contains records of interactions between Church and Kalmár before and after the publication of Kalmár's argument. Beyond the records presented here both Church's Nachlass at Princeton University and Kalmár's Nachlass at the Klebelsberg Library at the University of Szeged contain parts of their correspondence during the late 1940s and early 1950s concerning various issues about *The Journal of Symbolic Logic*. In addition, Church's Nachlass contains their correspondence between 1962 and 1964. However, that correspondence considers only Church's invitation to the Colloquium on the Foundations of Mathematics, Mathematical Machines and Their Applications, held at Tihany, Hungary, between 11–15 of September, 1962, organized by Kalmár. And issues of typesetting for Church's (1965) which was published in the proceedings of the Colloquium.

A picture showing Kalmár and Church, taken by András Ádám[31] at the Colloquium on the Foundations of Mathematics, Mathematical Machines and Their Applications, Tihany, 11–15, September, 1962.

[31] The picture is published with his approval.

People in the picture from left to right: unknown, Angéla Danóczi (Bólyai Society, head of administration), László Kalmár, unknown (in the background), János S. Petőfi (in the background, Hungarian Academy of Science, external member), Alonzo Church, Mrs. Church.

Kalmár applied for a full professor appointment at the University of Szeged in 1946 which he was appointed during 1947. One of Kalmár's Letter of Recommendation was written by Church:[32]

Princeton University
Princeton New Jersey

Department of
Mathematics

April 29, 1946

To Whom It may Concern:

This letter is written at the request of Dr. László Kalmár, for the purpose of giving an estimate of his standing as a mathematician and of the opinion which

[32]The other recommenders were John von Neumann and Haskell B. Curry.

is generally held of his original work by mathematicians in this country and elsewhere. In doing so, I confine myself to his research in mathematical logic, leaving it to others to supply an estimate of his contributions to other fields.

I have no hesitation in saying that the opinion of his work in mathematical logic – as held, not only by myself, but generally by those competent in this field – is of the highest. It is my judgment that his work in this field alone is sufficient to place him in a very high rank among living mathematicians.

Kalmár's work has been along the lines of Hilbertian proof theory, and he has made in particular a series of outstanding contributions to the important and difficult decision problem of the first-order functional calculus (or predicate calculus). For the present state of advancement of the work on this problem a large share of the credit must be given to Kalmár, both for his own results, and for the influence which his work has had in stimulating and serving as basis for contributions to the problem by others. Kalmár has also published significant papers on the foundations of the propositional calculus and on the axiomatic basis of definition by recursion. Unpublished work, known to me only in outline, includes also contributions to the question of consistency proofs and to the theory of recursive numerical functions. I predict with confidence that Kalmár may be expected to continue an outstanding original contributor to progress in the field of mathematical logic.

Kalmár has been since 1944 a consulting editor of The Journal of Symbolic Logic (of which I am one of the editors). The war, and the aftermath of the war, have so far prevented him from participating actively in the editorial work of the Journal, but I mention his appointment as an indication of the good opinion which we hold of him. (The board of consulting editors of the Journal is not large – there are at present eleven – and is carefully restricted to persons who in our judgement may be considered as outstanding in the field.)

signature of Alonzo Church
Alonzo Church
Associate Professor of Mathematics

(I cannot resist the temptation to share the following story of Martin Davis, told me in a private conversation,[33] linking Church and Kalmár indirectly. While Davis was working on his PhD at Princeton with Church as his advisor. Church

[33] Martin Davis gave me permission to share his story.

recommended to him a paper of Kalmár's that might be useful for his dissertation. When borrowing the item from the library, to Davis' surprise, it turned out to be in Hungarian. [Most likely it it was Kalmár's (1943)] However, the library did not have a Hungarian–English dictionary, only a Hungarian–Russian one. Therefore Davis ended up translating Kalmár's text first to Russian and then that to English.)

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